

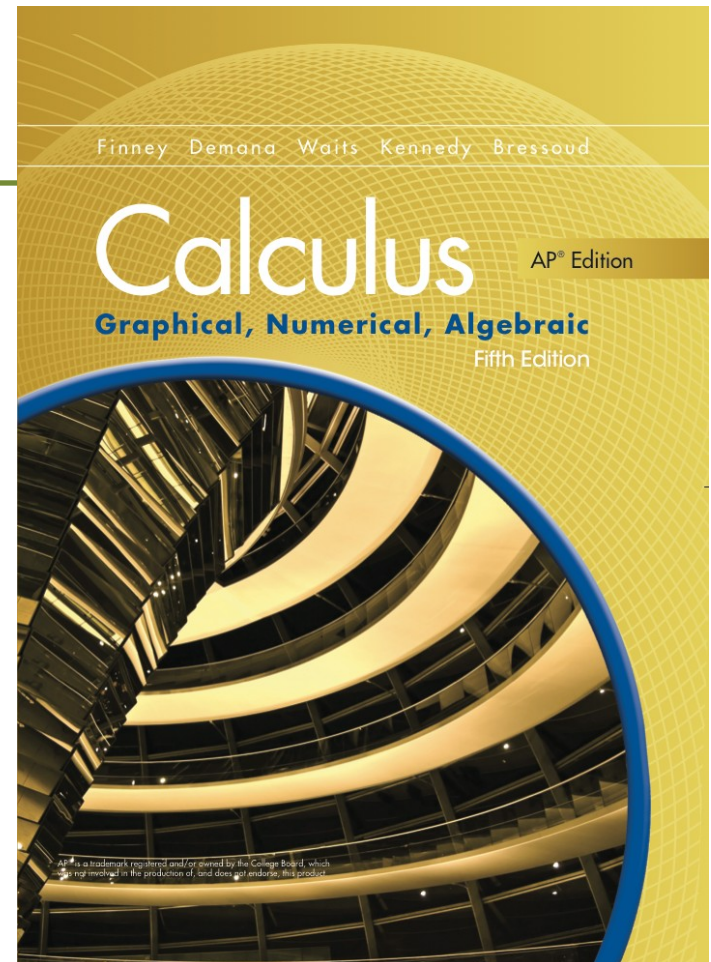
Chapter 10

Infinite Series



Section 10.3

Taylor's Theorem



Quick Review

Find the smallest number M that bounds $|f|$ from above on the interval I (that is, find the smallest M such that $|f(x)| \leq M$ for all x in I).

1. $f(x) = 2\cos(3x), \quad I = [-2\pi, 2\pi]$

2. $f(x) = x^2 + 3 \quad I = [1, 2]$

3. $f(x) = 2^x \quad I = [-3, 0]$

4. $f(x) = \frac{x}{x^2 + 1} \quad I = [-2, 2]$

Quick Review

Tell whether the function has derivatives of all orders at the given values of a .

5. $\frac{x}{x+1}$, $a=0$

6. $|x^2 - 4|$, $a=2$

7. $\sin x + \cos x$, $a=\pi$

8. e^x , $a=0$

9. $x^{\frac{3}{2}}$, $a=0$

Quick Review Solutions

Find the smallest number M that bounds $|f|$ from above on the interval I (that is, find the smallest M such that $|f(x)| \leq M$ for all x in I).

1. $f(x) = 2\cos(3x)$, $I = [-2\pi, 2\pi]$ 2

2. $f(x) = x^2 + 3$ $I = [1, 2]$ 7

3. $f(x) = 2^x$ $I = [-3, 0]$ 1

4. $f(x) = \frac{x}{x^2 + 1}$ $I = [-2, 2]$ 1/2

Quick Review Solutions

Tell whether the function has derivatives of all orders at the given values of a .

5. $\frac{x}{x+1}$, $a=0$ Yes

6. $|x^2 - 4|$, $a=2$ No

7. $\sin x + \cos x$, $a=\pi$ Yes

8. e^{-x} , $a=0$ Yes

9. $x^{\frac{3}{2}}$, $a=0$ No

What you'll learn about

- Taylor polynomials and truncation error
- Truncation error for a geometric series
- Truncation error bound for certain alternating series
- Taylor's Theorem with Remainder
- The Lagrange error bound
- The Remainder Bounding Theorem
- Deriving Euler's Formula from Maclaurin series

... and why

If we approximate a function represented by a power series by its Taylor polynomials, it is important to know how to determine the error in the approximation.

Example Approximating a Function to Specifications

Find a Taylor polynomial that will serve as an adequate substitute for $\sin x$ on the interval $[-\pi, \pi]$.

Choose $P_n(x)$ so that $|P_n(x) - \sin x| < 0.0001$ for every x in the interval $[-\pi, \pi]$. We need to make $|P_n(\pi) - \sin \pi| < 0.0001$, because then P_n will be adequate throughout the interval.

$$|P_n(\pi) - \sin \pi| < 0.0001$$

$$|P_n(\pi)| < 0.0001$$

Evaluate partial sums at $x = \pi$, adding one term at a time.

Eventually you will find the following:

$$\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \frac{\pi^9}{9!} - \frac{\pi^{11}}{11!} + \frac{\pi^{13}}{13!} = 2.114256749 \times 10^{-5}.$$

Truncation Error

In practical terms, then, we would like to be able to use Taylor polynomials to approximate functions over the intervals of convergence of the Taylor series, and we would like to keep the error of the approximation within specified bounds. Since the error results from *truncating* the series down to a polynomial (that is, cutting it off after some number of terms), we call it the **truncation error**.

Alternating Series Error Bound

Suppose the terms of a series have the following three properties:

1. the terms alternate in sign;
2. the terms decrease in absolute value;
3. the terms approach 0 as a limit.

Then the truncation error after n terms is less than the absolute value of the $(n + 1)$ st term.

Taylor's Theorem with Remainder

If f has derivatives of all orders in an open interval I containing a , then for each positive integer n and for each x in I

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x),$$

where $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ for some c between a and x

Taylor's Theorem

The first equation in Taylor's Theorem is **Taylor's** formula. The Function $R_n(x)$ is the **remainder of order n** or the **error term** for the approximation of f by $P_n(x)$ over I . It is also called the **Lagrange form of the remainder**, and bounds on $R_n(x)$ found using this form are **Lagrange error bounds**.

Example Proving Convergence of a Maclaurin Series

Prove that the series $\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$ converges to $\sin x$ for all real x .

Consider $R_n(x)$ as $n \rightarrow \infty$. By Taylor's Theorem, $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-0)^{n+1}$,

where $f^{(n+1)}(c)$ is the $(n+1)$ st derivative of $\sin x$ evaluated at some c between x and 0. For this function $-1 \leq f^{(n+1)}(c) \leq 1$ so,

$$|R_n(x)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!} (x-0)^{n+1} \right|$$

$$\leq \frac{1}{(n+1)!} |x^{n+1}| = \frac{|x^{n+1}|}{(n+1)!} \quad \text{As } n \rightarrow \infty, \text{ the factorial growth is larger in the}$$

denominator, than the exponential growth in the numerator. Therefore, as $n \rightarrow \infty$

$$\frac{|x^{n+1}|}{(n+1)!} \rightarrow 0 \text{ for all } x. \text{ This means that } R_n(x) \rightarrow 0 \text{ for all } x$$

Remainder Bounding Theorem

If there are positive constants M and r such that $|f^{(n+1)}(t)| \leq Mr^{n+1}$ for all t between a and x , then the remainder $R_n(x)$ in Taylor's Theorem

satisfies the inequality $|R_n(x)| \leq M \frac{r^{n+1} |x - a|^{n+1}}{(n+1)!}$.

If these conditions hold for every n and all the other conditions of Taylor's Theorem are satisfied by f , then the series converges to $f(x)$.

Example Proving Convergence

Use the Remainder Estimation Theorem to prove that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ for all real x .

We have already shown this to be the Taylor series generated by e^x at $x=0$.

We must verify $R_n(x) \rightarrow 0$ for all x . To do this, we must find M and r such that

$|f^{(n+1)}(t)| = e^t$ is bounded by Mr^{n+1} for t between 0 and arbitrary x .

Let M be the maximum value of e^t and let $r=1$.

If the interval is $[0, x]$, let $M = e^x$.

If the interval is $[x, 0]$, let $M = e^0 = 1$.

In either case, $e^t \leq M$ throughout the interval, and the Remainder Estimation Theorem guarantees convergence.

Euler's Formula

$$e^{ix} = \cos x + i \sin x$$

Quick Quiz Sections 10.1 – 10.3

1. Which of the following is the sum of the series $\sum_{n=0}^{\infty} \frac{\pi^n}{e^{2n}}$?

(A) $\frac{e}{e - \pi}$

(B) $\frac{\pi}{\pi - e}$

(C) $\frac{\pi}{\pi - e^2}$

(D) $\frac{e^2}{e^2 - \pi}$

(E) The series diverges

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Quick Quiz Sections 10.1 – 10.3

2. Assume that f has derivatives of all orders for all real numbers x , $f(0) = 2$, $f'(0) = -1$, $f''(0) = 6$, and $f'''(0) = 12$. Which of the following is the third order Taylor polynomial for f at $x = 0$?

- (A) $2 - x + 3x^2 + 2x^3$
- (B) $2 - x + 6x^2 + 12x^3$
- (C) $2 - \frac{1}{2}x + 3x^2 + 2x^3$
- (D) $-2 + x - 3x^2 - 2x^3$
- (E) $2 - x + 6x^2$

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Quick Quiz Sections 10.1 – 10.3

3. Which of the following is the Taylor series generated by $f(x) = 1/x$ at $x=1$?

(A) $\sum_{n=0}^{\infty} (x-1)^n$

(B) $\sum_{n=0}^{\infty} (-1)^n x^n$

(C) $\sum_{n=0}^{\infty} (-1)^n (x+1)^n$

(D) $\sum_{n=0}^{\infty} (-1)^n \frac{(x-1)^n}{n!}$

(E) $\sum_{n=0}^{\infty} (-1)^n (x-1)^n$

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